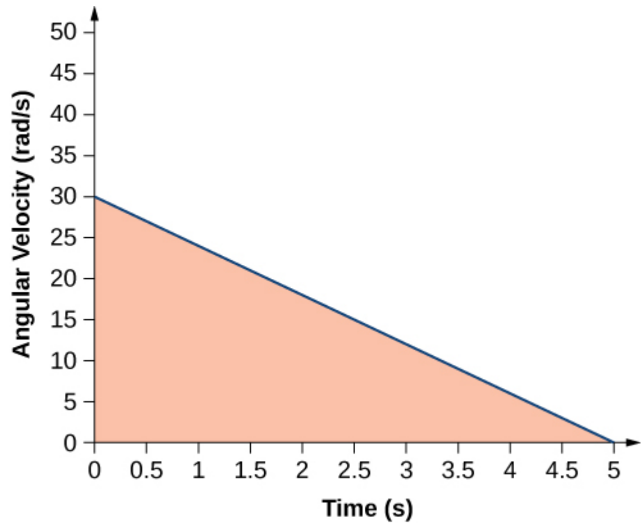

Figure 10.13 The area under the curve is the area of the right triangle.

$$\begin{aligned} \Delta\theta &= \text{area(triangle)}; \\ \Delta\theta &= \frac{1}{2}(30\text{ rad/s})(5\text{ s}) = 75\text{ rad.} \end{aligned}$$

We verify the solution using **Equation 10.12**:

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2.$$

Setting $\theta_0 = 0$, we have

$$\theta_0 = (30.0\text{ rad/s})(5.0\text{ s}) + \frac{1}{2}(-6.0\text{ rad/s}^2)(5.0\text{ rad/s})^2 = 150.0 - 75.0 = 75.0\text{ rad.}$$

This verifies the solution found from finding the area under the curve.

Significance

We see from part (b) that there are alternative approaches to analyzing fixed-axis rotation with constant acceleration. We started with a graphical approach and verified the solution using the rotational kinematic equations. Since $\alpha = \frac{d\omega}{dt}$, we could do the same graphical analysis on an angular acceleration-vs.-time curve. The area under an α-vs.-t curve gives us the change in angular velocity. Since the angular acceleration is constant in this section, this is a straightforward exercise.

10.3 | Relating Angular and Translational Quantities

Learning Objectives

By the end of this section, you will be able to:

- Given the linear kinematic equation, write the corresponding rotational kinematic equation
- Calculate the linear distances, velocities, and accelerations of points on a rotating system given the angular velocities and accelerations

In this section, we relate each of the rotational variables to the translational variables defined in **Motion Along a Straight Line** and **Motion in Two and Three Dimensions**. This will complete our ability to describe rigid-body rotations.

Angular vs. Linear Variables

In **Rotational Variables**, we introduced angular variables. If we compare the rotational definitions with the definitions of linear kinematic variables from **Motion Along a Straight Line** and **Motion in Two and Three Dimensions**, we find that there is a mapping of the linear variables to the rotational ones. Linear position, velocity, and acceleration have their rotational counterparts, as we can see when we write them side by side:

	Linear	Rotational
Position	x	θ
Velocity	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$

Let's compare the linear and rotational variables individually. The linear variable of position has physical units of meters, whereas the angular position variable has dimensionless units of radians, as can be seen from the definition of $\theta = \frac{s}{r}$, which is the ratio of two lengths. The linear velocity has units of m/s, and its counterpart, the angular velocity, has units of rad/s. In **Rotational Variables**, we saw in the case of circular motion that the linear tangential speed of a particle at a radius r from the axis of rotation is related to the angular velocity by the relation $v_{\text{t}} = r\omega$. This could also apply to points on a rigid body rotating about a fixed axis. Here, we consider only circular motion. In circular motion, both uniform and nonuniform, there exists a centripetal acceleration (**Motion in Two and Three Dimensions**). The centripetal acceleration vector points inward from the particle executing circular motion toward the axis of rotation. The derivation of the magnitude of the centripetal acceleration is given in **Motion in Two and Three Dimensions**. From that derivation, the magnitude of the centripetal acceleration was found to be

$$a_{\text{c}} = \frac{v_{\text{t}}^2}{r}, \quad (10.14)$$

where r is the radius of the circle.

Thus, in uniform circular motion when the angular velocity is constant and the angular acceleration is zero, we have a linear acceleration—that is, centripetal acceleration—since the tangential speed in **Equation 10.14** is a constant. If nonuniform circular motion is present, the rotating system has an angular acceleration, and we have both a linear centripetal acceleration that is changing (because v_{t} is changing) as well as a linear tangential acceleration. These relationships are shown in **Figure 10.14**, where we show the centripetal and tangential accelerations for uniform and nonuniform circular motion.

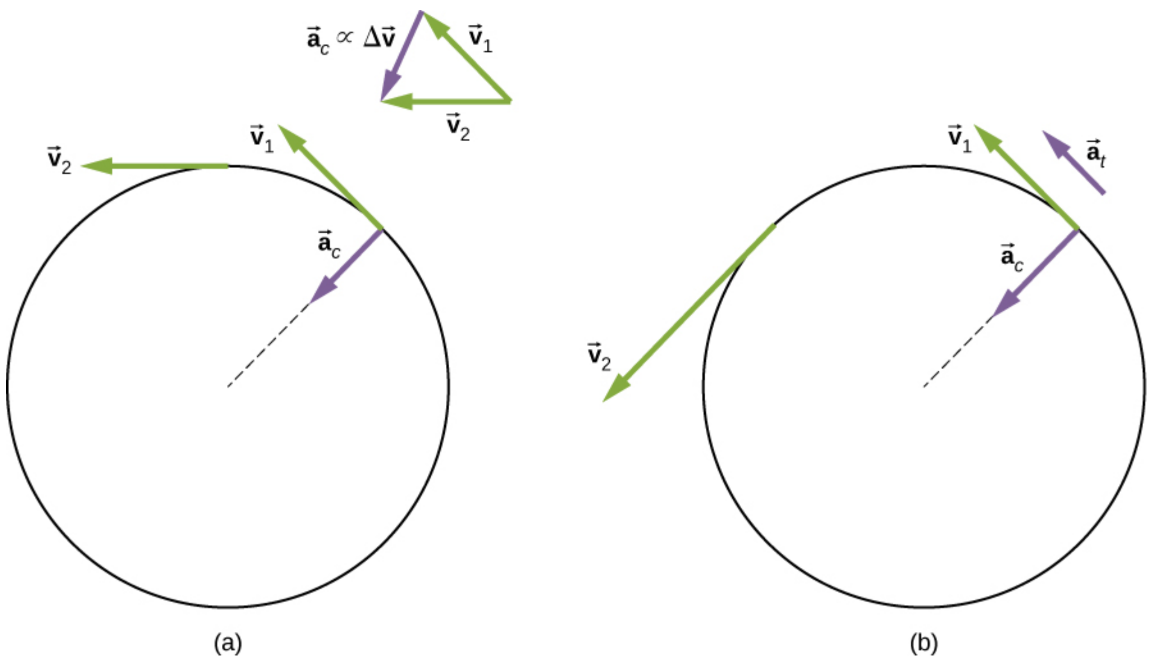

Figure 10.14 (a) Uniform circular motion: The centripetal acceleration a_c has its vector inward toward the axis of rotation. There is no tangential acceleration. (b) Nonuniform circular motion: An angular acceleration produces an inward centripetal acceleration that is changing in magnitude, plus a tangential acceleration a_t.

The centripetal acceleration is due to the change in the direction of tangential velocity, whereas the tangential acceleration is due to any change in the magnitude of the tangential velocity. The tangential and centripetal acceleration vectors $\vec{\mathbf{a}}_t$ and $\vec{\mathbf{a}}_c$ are always perpendicular to each other, as seen in **Figure 10.14**. To complete this description, we can assign a **total linear acceleration** vector to a point on a rotating rigid body or a particle executing circular motion at a radius *r* from a fixed axis. The total linear acceleration vector $\vec{\mathbf{a}}$ is the vector sum of the centripetal and tangential accelerations,

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_c + \vec{\mathbf{a}}_t. \quad (10.15)$$

The total linear acceleration vector in the case of nonuniform circular motion points at an angle between the centripetal and tangential acceleration vectors, as shown in **Figure 10.15**. Since $\vec{\mathbf{a}}_c \perp \vec{\mathbf{a}}_t$, the magnitude of the total linear acceleration is

$$|\vec{\mathbf{a}}| = \sqrt{a_c^2 + a_t^2}.$$

Note that if the angular acceleration is zero, the total linear acceleration is equal to the centripetal acceleration.

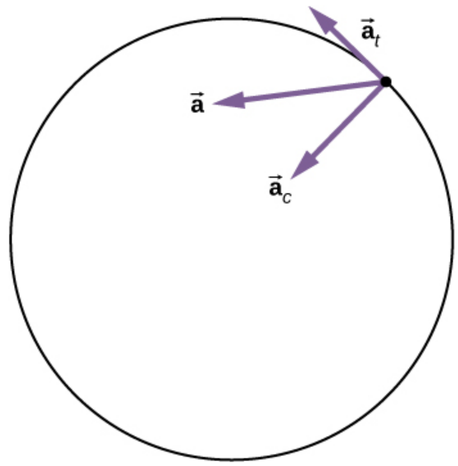

Figure 10.15 A particle is executing circular motion and has an angular acceleration. The total linear acceleration of the particle is the vector sum of the centripetal acceleration and tangential acceleration vectors. The total linear acceleration vector is at an angle in between the centripetal and tangential accelerations.

Relationships between Rotational and Translational Motion

We can look at two relationships between rotational and translational motion.

1. Generally speaking, the linear kinematic equations have their rotational counterparts. **Table 10.2** lists the four linear kinematic equations and the corresponding rotational counterpart. The two sets of equations look similar to each other, but describe two different physical situations, that is, rotation and translation.

Rotational	Translational
$\theta_f = \theta_0 + \bar{\omega} t$	$x = x_0 + \bar{v} t$
$\omega_f = \omega_0 + \alpha t$	$v_f = v_0 + at$
$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x_f = x_0 + v_0 t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$	$v_f^2 = v_0^2 + 2a(\Delta x)$

Table 10.2 Rotational and Translational Kinematic Equations

2. The second correspondence has to do with relating linear and rotational variables in the special case of circular motion. This is shown in **Table 10.3**, where in the third column, we have listed the connecting equation that relates the linear variable to the rotational variable. The rotational variables of angular velocity and acceleration have subscripts that indicate their definition in circular motion.

Rotational	Translational	Relationship (r = radius)
θ	s	$\theta = \frac{s}{r}$
ω	v_t	$\omega = \frac{v_t}{r}$
α	a_t	$\alpha = \frac{a_t}{r}$

Table 10.3 Rotational and Translational Quantities: Circular Motion

Rotational	Translational	Relationship (r = radius)
	a_c	$a_c = \frac{v_t^2}{r}$

Table 10.3 Rotational and Translational Quantities: Circular Motion

Example 10.7

Linear Acceleration of a Centrifuge

A centrifuge has a radius of 20 cm and accelerates from a maximum rotation rate of 10,000 rpm to rest in 30 seconds under a constant angular acceleration. It is rotating counterclockwise. What is the magnitude of the total acceleration of a point at the tip of the centrifuge at $t = 29.0\text{s}$? What is the direction of the total acceleration vector?

Strategy

With the information given, we can calculate the angular acceleration, which then will allow us to find the tangential acceleration. We can find the centripetal acceleration at $t = 0$ by calculating the tangential speed at this time. With the magnitudes of the accelerations, we can calculate the total linear acceleration. From the description of the rotation in the problem, we can sketch the direction of the total acceleration vector.

Solution

The angular acceleration is

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - (1.0 \times 10^4)2\pi/60.0\text{ s(rad/s)}}{30.0\text{ s}} = -34.9\text{ rad/s}^2.$$

Therefore, the tangential acceleration is

$$a_t = r\alpha = 0.2\text{ m}(-34.9\text{ rad/s}^2) = -7.0\text{ m/s}^2.$$

The angular velocity at $t = 29.0\text{ s}$ is

$$\begin{aligned}\omega &= \omega_0 + \alpha t = 1.0 \times 10^4\left(\frac{2\pi}{60.0\text{ s}}\right) + \left(-34.9\text{ rad/s}^2\right)(29.0\text{ s}) \\ &= 1047.2\text{ rad/s} - 1012.71 = 35.1\text{ rad/s}.\end{aligned}$$

Thus, the tangential speed at $t = 29.0\text{ s}$ is

$$v_t = r\omega = 0.2\text{ m}(35.1\text{ rad/s}) = 7.0\text{ m/s}.$$

We can now calculate the centripetal acceleration at $t = 29.0\text{ s}$:

$$a_c = \frac{v^2}{r} = \frac{(7.0\text{ m/s})^2}{0.2\text{ m}} = 245.0\text{ m/s}^2.$$

Since the two acceleration vectors are perpendicular to each other, the magnitude of the total linear acceleration is

$$\left|\vec{\mathbf{a}}\right| = \sqrt{a_c^2 + a_t^2} = \sqrt{(245.0)^2 + (-7.0)^2} = 245.1\text{ m/s}^2.$$

Since the centrifuge has a negative angular acceleration, it is slowing down. The total acceleration vector is as shown in **Figure 10.16**. The angle with respect to the centripetal acceleration vector is

$$\theta = \tan^{-1}\frac{-7.0}{245.0} = -1.6°.$$

The negative sign means that the total acceleration vector is angled toward the clockwise direction.

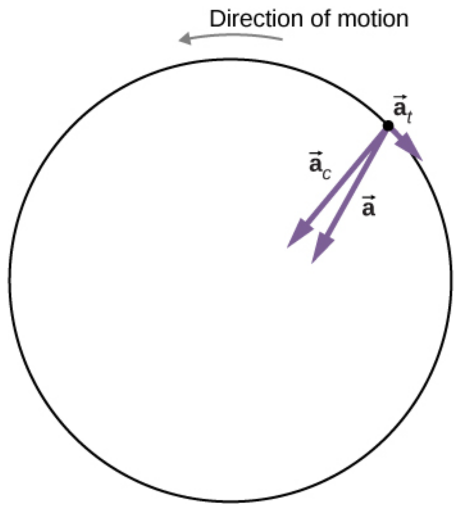

Figure 10.16 The centripetal, tangential, and total acceleration vectors. The centrifuge is slowing down, so the tangential acceleration is clockwise, opposite the direction of rotation (counterclockwise).

Significance

From **Figure 10.16**, we see that the tangential acceleration vector is opposite the direction of rotation. The magnitude of the tangential acceleration is much smaller than the centripetal acceleration, so the total linear acceleration vector will make a very small angle with respect to the centripetal acceleration vector.

10.3 Check Your Understanding A boy jumps on a merry-go-round with a radius of 5 m that is at rest. It starts accelerating at a constant rate up to an angular velocity of 5 rad/s in 20 seconds. What is the distance travelled by the boy?

Check out this **PhET simulation (https://openstaxcollege.org/l/21rotatingdisk)** to change the parameters of a rotating disk (the initial angle, angular velocity, and angular acceleration), and place bugs at different radial distances from the axis. The simulation then lets you explore how circular motion relates to the bugs' *xy*-position, velocity, and acceleration using vectors or graphs.

10.4 | Moment of Inertia and Rotational Kinetic Energy

Learning Objectives

By the end of this section, you will be able to:

- Describe the differences between rotational and translational kinetic energy
- Define the physical concept of moment of inertia in terms of the mass distribution from the rotational axis
- Explain how the moment of inertia of rigid bodies affects their rotational kinetic energy
- Use conservation of mechanical energy to analyze systems undergoing both rotation and translation
- Calculate the angular velocity of a rotating system when there are energy losses due to nonconservative forces

So far in this chapter, we have been working with rotational kinematics: the description of motion for a rotating rigid body